EFFICIENT DISTANCE ESTIMATION FOR FITTING IMPLICIT QUADRIC SURFACES

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ABSTRACT

This paper presents a novel approach for estimating the shortest Euclidean distance from a given point to the corresponding implicit quadric fitting surface. It first estimates the orthogonal orientation to the surface from the given point; then the shortest distance is directly estimated by intersecting the implicit surface with a line passing through the given point according to the estimated orthogonal orientation. The proposed orthogonal distance estimation is easily obtained without increasing computational complexity; hence it can be used in error minimization surface fitting frameworks. Comparisons of the proposed metric with previous approaches are provided to show both improvements in CPU time as well as in the accuracy of the obtained results. Surfaces fitted by using the proposed geometric distance estimation and state of the art metrics are presented to show the viability of the proposed approach.

1. INTRODUCTION

Having an efficient approximation to the shortest distance between a point and the fitting surface has been an important problem in different applications: reverse engineering [8], visualization [9], object recognition [4]. In the context of 3D computer vision the quadric surface fitting has been an active research topic during the last two decades [1, 7]. The appearance of new sensors, which allow to obtain a large amount of 3D data in a reduced time, and the need to process all this information efficiently have opened new challenges looking for efficient fitting approaches.

In general, fitting approaches are based on finding the set of parameters that minimize some distance measures between the given set of points and the fitted curve or surface. Unfortunately, the minimization of the Euclidean distance from the data points to a general curve or surface is computationally impractical, since there is no closed form expression. Hence, the shortest Euclidean distance has been approximated and

different criteria have been proposed in the literature for estimating this residual value and judge goodness of fit (e.g., Algebraic distance [2], PCA-based normal vector estimation [6]).

The current work proposes a novel technique for an efficient estimation of the shortest Euclidean distance, hereinafter orthogonal distance, from a given point p to the corresponding implicit quadric fitting surface, $F(\mathbf{b}, \mathbf{X}) = 0$. Implicit quadric are useful in the computer vision field for representing 2D or 3D data in a compact way. The advantage of the proposed error estimation is twofold. First, it provides a more accurate value than current approaches. Second, it can be efficiently computed and run in real time. The rest of the paper is organized as follows. Section 2 describes the problem and introduces related work. The proposed technique is presented in section 3. Section 4 gives experimental results and comparisons.

2. RELATED WORK

Computing the orthogonal distance between a point p and an implicit general quadric:

$$F(\mathbf{b}, \mathbf{X}) = b_1 x^2 + b_2 y^2 + b_3 z^2 + 2b_4 xy + 2b_5 xz \qquad (1)$$
$$+ 2b_6 yz + 2b_7 x + 2b_8 y + 2b_9 z + 1 = 0$$

can be formulated as first finding the closest point p_0 on the surface. This point corresponds to the intersection of the surface with a line passing through p and orthogonal to the surface at p_0, $\overrightarrow{n}_0 \equiv (a_0, b_0, c_0)$; thus, the orthogonal distance can be expressed as $p - p_0 = \alpha \overrightarrow{n}_0$, for some scalar α:

$$\frac{x_p - x_0}{a_0} = \frac{y_p - y_0}{b_0} = \frac{z_p - z_0}{c_0} \qquad (2)$$

with $F(\mathbf{b}, \mathbf{X}) = 0$. However, since $\overrightarrow{n}_0$ is also unknown, the problem requires solving a non-linear equation.

In order to overcome the previous problem—i.e., solving a non-linear equation for every point to be fitted—different approximations have been proposed. For instance, [3] proposes to intersect the quadric surface with three lines, one
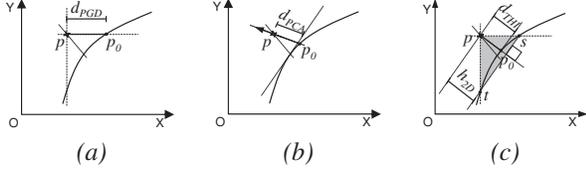
Fig. 1. Orthogonal distance estimation between $p_{(x,y)}$ and a quadric curve. (a) The shortest distance is selected, Pseudo Geometric Distance (PGD) [3]; (b) By using an estimation of the surface orientation (PCA in a local neighborhood) [6]. (c) Distance estimation based on the proposed approach (i.e., by finding $p_{0(x,y)}$ from the Triangle Height Intersection (THI)).
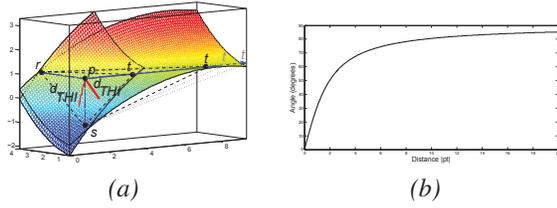

Fig. 2. Transition from a tetrahedron height vector to a triangle height vector. (a) Illustration of two surfaces with their corresponding intersection points (planar triangular patch (r, s, t)). (b) Angle between tetrahedron height segment and (pt) segment, as a function of distance $|pt|$.

parallel to each of the X, Y, and Z axes (see illustration[1] in Fig. 1 (a)). Hence, vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are considered as approximations for $\overrightarrow{n}_0$. For each one of these approximations one quadratic equation with one unknown $F(x, y_p, z_p) = 0$, $F(x_p, y, z_p) = 0$ and $F(x_p, y_p, z) = 0$ should be solved; giving rise at most six solutions (two roots per quadratic equation), and six distances. The minimum distance is taken as the residual value and referred as Pseudo Geometric Distance (PGD).

A more accurate orthogonal distance estimation has been presented in [6] by first estimating a surface normal vector for each point p. These vectors are computed by using principal components analysis (PCA) in a small $N \times N$ neighborhood centered at each point. In other words, $\overrightarrow{n}_p$ is defined as the eigenvector of the local covariance matrix associated with the smallest eigenvalue. A consistency step is applied to remove outliers. In case p is an inlier, the surface is intersected with a line passing through p and parallel to $\overrightarrow{n}_p$, resulting in a better approximation for the closest point p_0. A single quadratic equation $F(\mathbf{b}, \mathbf{X}) = 0$ should be solved; from the two possible roots, two distances result and the smallest is taken as the orthogonal distance estimation (Fig. 1 (b) shows an illustration of this technique).

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3. PROPOSED APPROACH

The proposed approach consists in approximating the surface normal vector $\overrightarrow{n}_0 \equiv (a_0, b_0, c_0)$, at the intersection point p_0, by means of a vector $\overrightarrow{h} \equiv (a_h, b_h, c_h)$ parallel to the tetrahedron height segment; the tetrahedron is defined by the points: p, $F(x_p, y, z) = 0$, $F(x, y_p, z) = 0$ and $F(x, y, z_p) = 0$ (see an illustration for the 2D case in Fig. 1 (c)). A faster and coarser estimation of the orthogonal distance could be easily obtained from the tetrahedron height segment. In the particular case we are tackling in this work, since the fitted curve is defined by an implicit quadric equation $F(\mathbf{b}, \mathbf{X}) = 0$, the intersection point can be found by representing the tetrahedron height by means of a parametric equation:

$$x = x_p + a_h u, \quad y = y_p + b_h u, \quad z = z_p + c_h u \ , \qquad (3)$$

then, by replacing (3) in the implicit quadric expression, and by solving that quadric equation the two values satisfying the implicit equation are found. The nearest one correspond to the intersection of the tetrahedron height segment and the implicit quadric curve (i.e., p_0); $|p - p_0|$ is the orthogonal distance estimation resulting from the proposed approach (referred as THI hereinafter).

Let r, s and t be the three intersections with a quadric surface, which create a triangular planar patch (see Fig. 2 (a)). Vectors $\overrightarrow{rs}$ and $\overrightarrow{rt}$ are contained in the plane and their cross product is orthogonal to the planar patch; in other words it is parallel to the tetrahedron height segment $\overrightarrow{h} \equiv (a_h, b_h, c_h)$, which is used as an estimation of the surface normal vector $\overrightarrow{n}_0 \equiv (a_0, b_0, c_0)$, at the intersection point p_0.

Note that in the extreme cases, when intersections with some of the directions $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ cannot be found the 3D case becomes into: $i)$ the 2D case (two intersections); $ii)$ the PGD case (only one intersection); or $iii)$ the point p is an outlier since none of the three directions intersects the implicit quadric surface. Transitions between different cases are smoothly reached; Fig. 2 (a) shows an illustration where one of the vertices of the triangular patch (r, s, t) moves away from current position up to the extreme—i.e., no intersection between vertex t and the implicit surface can be found[2]; the smooth transition from the tetrahedron height segment orientation to the triangle height segment orientation can be appreciated in the illustration of Fig. 2 (b).

4. EXPERIMENTAL RESULTS

This section provides results obtained with synthetic and real data points. Synthetic data points are used as ground truth for evaluating results of the proposed approach when noisy data are considered. Real data points have been obtained with the

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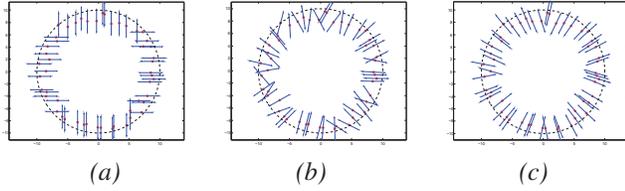
(a)   *(b)*   *(c)*

Fig. 3. Estimated normal vectors for raw data points by using different approaches. (*a*) PGD approach. (*b*) Based on PCA in a local neighborhood. (*c*) Proposed approach (THI direction).

Fig. 4. Accumulated error by using different approaches as a function of noise standard deviation.

K2T structured light camera system at the University of South Florida and are used for validating the proposed approach.

    Synthetic Data: In this section results from 2D and 3D synthetic data points are presented. The first illustration correspond to a set of 50 noisy 2D data points, randomly distributed over a circle. Fig. 3 presents normal vectors $\overrightarrow{n}_0$ estimated by using different approaches. An outer, and concentric, circle is just depicted for highlighting these orientations. Fig. 3 (*a*) shows the orientations obtained by using the PGD criterion [3]; since the shortest distance along $\{(1,0),(0,1)\}$ is selected, the estimated $\overrightarrow{n}_0$ is parallel to one of the axes. Fig. 3 (*b*) presents normal vectors estimated by using the approach presented in [6]. Since this approach is based on a PCA analysis in a small neighborhood centered at each point, it is easily affected by noisy data. It can be appreciated that some estimations are completely wrong. On the contrary, results obtained with the proposed approach are robust to noisy data (see Fig. 3 (*c*)). In most of the cases estimated normal vectors are correctly oriented towards the center of the circle.

    Fig. 4(*top*) presents results obtained with a set of 3D data points uniformly distributed over the surface of a sphere. An accumulated real orthogonal distance (ROD), from the whole set of 3D data points to an outer and concentric sphere, is used as ground truth. Estimated distances, also accumulated over the whole set of points, are computed by using the three different approaches (i.e., PGD [3], PCA [6] and THI). The experiment has been repeated by increasing the standard deviation of the noise added to the set of 3D points. It can be seen that the proposed approach behaves quite similar to the real orthogonal distance independently of the added noise. On the contrary, as it was expected, results obtained with the PCA based approach are easily biased when the noise standard deviation grows. Similar results have been obtained by using 3D data points uniformly distributed over the surface of a cylinder (Fig. 4(*bottom*)). In this case, distances to an outer and concentric cylinder are considered. The real orthogonal distance, accumulated through the whole set of points, is used as ground truth. It can be appreciated that the distances estimated with the proposed approach (THI) are very similar to the real values; on the other hand values computed with the proposed approach are not affected by noisy data.

    Real Data: The objective of this section is to show the viability of the proposed approach by analyzing surfaces fitted
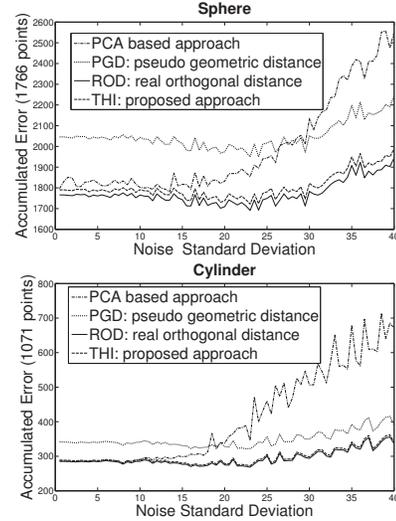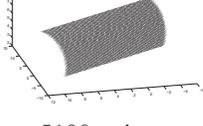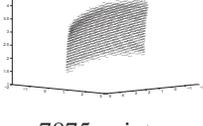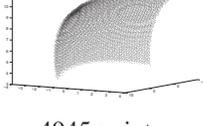
using different metrics. A simple RANSAC-based [5] fitting approach has been used for finding the best set of quadric surface parameters in each case; developing a novel fitting technique is out of the scope of current work. The prior knowledge about the geometry of the surface to be fitted and a 3D data preprocessing step are used to reduce the size of parameter vector **b** in (1). The preprocessing step consists in rotating and translating the set of 3D data points in order to place the world coordinate system at the centroid of the set of points and to orient it according to the main axis of the 3D data. Once data have been transformed to the new coordinate system the RANSAC-based fitting approach is used for computing the surface parameters (parameter vector[3] **b**).

    Table 1 shows the original images (left column) as well as the patches extracted for fitting (second column) that only contains 3D data points from a quadric surface. Surface parameters have been obtained through the RANSAC-based fitting by using three different estimations of the orthogonal distance. Results are presented on columns 3 to 5 of Table 1. The fastest results were obtained by using the PGD estimation; however, as presented in the synthetic case, this distance estimation is not accurate and could leads to a biased result. Note that in the real data case it is not possible to judge the goodness of obtained results since there are not ground truth values for the surface parameters. In spite of that, it can be appreciated that in most of the cases the surface parameters obtained with PCA based approach are very similar to the ones computed using the proposed approach. In other words, it could be assumed that PGD based approach results in a more inaccurate set of surface parameters. Finally, although the proposed approach gives similar results to the PCA based ap-

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[3]It is defined according to the kind of surface to be fitted—e.g., cylinder: ($\mathbf{b} = \{b_2 = b_3, b_8, b_9\}$).

Table 1. Surfaces fitted by using different metrics; Rad: radius; C: center; AEr: Accumulated Error according to the used metric; CPU time in sec.

Intensity Image	3D Data to be Fitted	PGD [3]	PCA [6]	THI: Prop. Ap.
	5100 points	Rad = 4.286 C = (0.08, 3.92) AEr = 805.27 CPU = 25.378	Rad = 4.063 C = (-0.07, 3.64) AEr = 765.45 CPU = 310.391	Rad = 4.091 C = (-0.06, 3.68) AEr = 685.47 CPU = 29.14
	7875 points	Rad = 3.269 C = (2.4, 2.1, 1.4) AEr = 1.40e+3 CPU = 19.596	Rad = 3.184 C = (2.4, 1.9, 1.4) AEr = 1.65e+3 CPU = 1013.415	Rad = 3.359 C = (2.4, 1.9, 1.2) AEr = 1.05e+3 CPU = 87.784
	4945 points	Rad = 7.168 C = (3.7, 3.1, 4.7) AEr = 757.35 CPU = 18.203	Rad = 6.457 C = (3.0, 2.9, 4.9) AEr = 653.31 CPU = 429.6	Rad = 6.751 C = (3.4, 3.0, 4.9) AEr= 573.98 CPU = 62.537
	7700 points	Rad = 10.132 C = (0.6, 2.0, 0.2) AEr = 1.12e+3 CPU = 28.184	Rad = 10.805 C = (0.02, 1.5, 0.1) AEr = 963.364 CPU = 1142.977	Rad = 10.289 C = (0.4, 1.7, 0.4) AEr = 862.88 CPU = 103.046
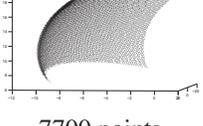
proach, it should be highlighted that it runs up to eleven times faster. Furthermore, recall that PCA based approach needs the right tuning for the neighbor point selection.

5. CONCLUSIONS

This paper presents an efficient approach for orthogonal distance estimation to be used in the implicit quadric surface fitting problem. It does not require any predefined adjustment. It is based on an estimation of a vector orthogonal to the surface for a given point. The intersection of a line with the implicit surface—line passing through the given point and parallel to the estimated vector—is defined as the orthogonal distance estimation. Experiments on synthetic and real data have shown the accuracy and robustness of the proposed approach. Furthermore, it is considerable faster than approaches with similar accuracy. Although the development of a surface fitting algorithm has not be addressed in the current paper, it is expected that the proposed metric can be useful in a minimization based context.

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